

CONSTANT CURVATURE FACTORABLE SURFACES IN 3-DIMENSIONAL ISOTROPIC SPACE

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ABSTRACT. In this paper, we study factorable surfaces in a 3-dimensional isotropic space. We classify such surfaces with constant isotropic Gaussian (K) and mean curvature (H). We provide a non-existence result related with the surfaces satisfying $\frac{H}{K} = \text{const.}$ Several examples are also illustrated.

1. INTRODUCTION

Let \mathbb{E}^3 be a 3-dimensional Euclidean space and (x, y, z) rectangular coordinates. A surface in \mathbb{E}^3 is said to be *factorable* (so-called *homothetical*) if it is a graph surface associated with $z = f(x)g(y)$ (see [4, 9]).

Constant Gaussian (K) and mean curvature (H) factorable surfaces in \mathbb{E}^3 were obtained in [8, 9, 16]. As more general case, Zong et al. [17] defined that an *affine factorable surface* in \mathbb{E}^3 is the graph of the function

$$z = f(x)g(y + ax), \quad a \neq 0$$

and classified ones with K, H constants.

In a 3-dimensional Minkowski space \mathbb{E}_1^3 , a surface is said to be *factorable* if it can be expressed by one of the explicit forms:

$$\Phi_1 : z = f(x)g(y), \quad \Phi_2 : y = f(x)g(z), \quad \Phi_3 : x = f(y)g(z).$$

Six different classes of the factorable surfaces in \mathbb{E}_1^3 appear with respect to the causal characters of the directions (for details, see [10]). Such surfaces of K, H constants in \mathbb{E}_1^3 were described in [7, 10, 15].

Besides the Minkowskian space, a 3-dimensional isotropic space \mathbb{I}^3 provides two different types of the factorable surfaces. It is indeed a product of the xy -plane and the isotropic z -direction with degenerate metric (cf. [5]). Due to the isotropic axes in \mathbb{I}^3 the factorable surface Φ_1 distinctly behaves from others. We call it the *factorable surface of type 1*. We refer to [1]-[3] for its details in \mathbb{I}^3 .

The factorable surfaces Φ_2, Φ_3 in \mathbb{I}^3 are locally isometric and, up to a sign, have same the second fundamental form. This means to have same isotropic Gaussian K and, up to a sign, mean curvature H . These surfaces are said to be of *type 2*.

In this manner we are mainly interested with the factorable surfaces of type 2 in \mathbb{I}^3 . We describe such surfaces in \mathbb{I}^3 with $K, H, H/K$ constants by the following results:

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Theorem 1.1. *A factorable surface of type 2 (Φ_3) in \mathbb{I}^3 has constant isotropic mean curvature H_0 if and only if, up to suitable translations and constants, one of the following occurs:*

- (i) *If Φ_3 is isotropic minimal, i.e. $H_0 = 0$;*
 - (i.1) *Φ_3 is a non-isotropic plane,*
 - (i.2) *$x = y \tan(cz)$,*
 - (i.3) *$x = c \frac{z}{y}$.*
- (ii) *Otherwise ($H_0 \neq 0$), $x = \pm \sqrt{\frac{-z}{H_0}}$,*

where c is some nonzero constant.

Theorem 1.2. *A factorable surface of type 2 (Φ_3) in \mathbb{I}^3 has constant isotropic Gaussian curvature K_0 if and only if, up to suitable translations and constants, one of the following holds:*

- (i) *If Φ_3 is isotropic flat, i.e. $K_0 = 0$;*
 - (i.1) *$x = c_1 g(z)$, $g' \neq 0$,*
 - (i.2) *$x = c_1 e^{c_2 y + c_3 z}$,*
 - (i.3) *$x = c_1 y^{c_2} z^{c_3}$, $c_2 + c_3 = 1$.*
- (ii) *Otherwise ($K_0 \neq 0$);*
 - (ii.1) *K_0 is negative and $x = \pm \sqrt{\frac{z}{-K_0}} y$,*
 - (ii.2)

$$f(y) = \frac{c_1}{y} \text{ and } z = \pm \int \left(c_2 g^{-1} - \frac{K_0}{c_1^2} \right)^{-1/2} dg,$$

where c_1, c_2, c_3 are some nonzero constants.

Theorem 1.3. *There does not exist a factorable surface of type 2 in \mathbb{I}^3 that satisfies $H/K = \text{const.} \neq 0$.*

We remark that the results are also valid for the factorable surface Φ_2 in \mathbb{I}^3 by replacing x with y as well as taking $y = \pm \sqrt{z/H_0}$ in the last statement of Theorem 1.1.

2. PRELIMINARIES

For detailed properties of isotropic spaces, see [6], [11]-[14].

Let $P(\mathbb{R}^3)$ be a real 3-dimensional projective space and ω a plane in $P(\mathbb{R}^3)$. Then $P(\mathbb{R}^3) \setminus \omega$ becomes a real 3-dimensional affine space. Denote $(x_0 : x_1 : x_2 : x_3) \neq (0 : 0 : 0 : 0)$ the projective coordinates in $P(\mathbb{R}^3)$.

A 3-dimensional *isotropic space* \mathbb{I}^3 is an affine space whose the absolute figure consists of a plane (absolute plane) ω and complex-conjugate straight lines (absolute lines) l_1, l_2 in ω . In coordinate form, ω is given by $x_0 = 0$ and l_1, l_2 by $x_0 = x_1 \pm ix_2 = 0$. The absolute point, $(0 : 0 : 0 : 1)$, is the intersection of the absolute lines.

For $x_0 \neq 0$, we have the affine coordinates by $x = \frac{x_1}{x_0}$, $y = \frac{x_2}{x_0}$, $z = \frac{x_3}{x_0}$. The group of motions of \mathbb{I}^3 is given by

$$(2.1) \quad (x, y, z) \mapsto (x', y', z') : \begin{cases} x' = a_1 + x \cos \phi - y \sin \phi, \\ y' = a_2 + x \sin \phi + y \cos \phi, \\ z' = a_3 + a_4 x + a_5 y + z, \end{cases}$$

where $a_1, \dots, a_5, \phi \in \mathbb{R}$.

The *isotropic metric* that is an invariant of (2.1) is induced by the absolute figure, namely $ds^2 = dx^2 + dy^2$. One is degenerate along the lines in z -direction and these lines are said to be *isotropic*. A plane is said to be *isotropic* if it involves an isotropic line. Otherwise it is called *non-isotropic plane* or *Euclidean plane*.

We restrict our framework to regular surfaces whose the tangent plane at each point is Euclidean, namely *admissible surfaces*.

Let M be a regular admissible surface in \mathbb{I}^3 locally parameterized by

$$r(u, v) = (x(u, v), y(u, v), z(u, v))$$

for a coordinate pair (u, v) . The components E, F, G of the first fundamental form of M in \mathbb{I}^3 are computed by the induced metric from \mathbb{I}^3 . The unit normal vector of M is the unit vector parallel to the z -direction.

The components of the second fundamental form II of M are given by

$$l = \frac{\det(r_{uu}, r_u, r_v)}{\sqrt{EG - F^2}}, \quad m = \frac{\det(r_{uv}, r_u, r_v)}{\sqrt{EG - F^2}}, \quad n = \frac{\det(r_{vv}, r_u, r_v)}{\sqrt{EG - F^2}}.$$

Accordingly, the *isotropic Gaussian* (or *relative*) and *mean curvature* of M are respectively defined by

$$K = \frac{ln - m^2}{EG - F^2}, \quad H = \frac{En - 2Fm + Gl}{2(EG - F^2)}.$$

A surface in \mathbb{I}^3 is said to be *isotropic minimal* (*isotropic flat*) if H (K) vanishes.

3. PROOF OF THEOREM 1.1

A factorable surface of type 2 in \mathbb{I}^3 can be locally expressed by either

$$\Phi_2 : r(x, z) = (x, f(x)g(z), z) \text{ or } \Phi_3 : r(y, z) = (f(y)g(z), y, z).$$

All over this paper, all calculations shall be done for the surface Φ_3 . Its first fundamental form in \mathbb{I}^3 turns to

$$ds^2 = (1 + (f'g)^2) dy^2 + 2(fgf'g') dydz + (fg')^2 dz^2,$$

where $f' = \frac{df}{dy}$, $g' = \frac{dg}{dz}$. Note that g' must be nonzero to obtain a regular admissible surface. By a calculation for the second fundamental form of Φ_3 we have

$$II = \left(\frac{f''g}{fg'} \right) dy^2 + 2 \left(\frac{f'}{f} \right) dydz + \left(\frac{g''}{g'} \right) dz^2, \quad g' \neq 0.$$

Therefore the isotropic mean curvature H of Φ_3 becomes

$$(3.1) \quad H = \frac{\left((f'g)^2 + 1 \right) g'' + g(g')^2 (ff'' - 2(f')^2)}{2f^2(g')^3}.$$

Let us assume that $H = H_0 = \text{const}$. First we distinguish the case in which Φ_3 is isotropic minimal:

Case A. $H_0 = 0$. (3.1) reduces to

$$(3.2) \quad \left((f'g)^2 + 1 \right) g'' + g(g')^2 (ff'' - 2(f')^2) = 0.$$

Case A.1. $f = f_0 \neq 0 \in \mathbb{R}$. (3.2) immediately implies $g = c_1 z + c_2$, $c_1, c_2 \in \mathbb{R}$, and thus we deduce that Φ_3 is a non-isotropic plane. This gives the statement (i.1) of Theorem 1.1.

Case A.2. $f = c_1y + c_2$, $c_1, c_2 \in \mathbb{R}$, $c_1 \neq 0$. (3.2) turns to

$$\frac{g''}{g'} = \frac{2c_1^2 g g'}{1 + (c_1 g)^2}.$$

By solving this one, we obtain

$$g = \frac{1}{c_1} \tan(c_2 z + c_3), \quad c_2, c_3 \in \mathbb{R}, \quad c_2 \neq 0,$$

which proves the statement (i.2) of Theorem 1.1.

Case A.3. $f'' \neq 0$. By dividing (3.2) with $g(g')^2$ one can be rewritten as

$$(3.3) \quad \left((f'g)^2 + 1 \right) \frac{g''}{g(g')^2} + f f'' - 2(f')^2 = 0.$$

Taking partial derivative of (3.3) with respect to z and after dividing with $(f')^2$, we get

$$(3.4) \quad 2 \frac{g''}{g'} + \left(\frac{1}{(f')^2} + g^2 \right) \left(\frac{g''}{g(g')^2} \right)' = 0.$$

By taking partial derivative of (3.4) with respect to y , we find $g'' = c_1 g (g')^2$, $c_1 \in \mathbb{R}$. We have two cases:

Case A.3.1. $c_1 = 0$. (3.3) reduces to

$$f f'' - 2(f')^2 = 0$$

and by solving it we derive

$$f = -\frac{1}{c_2 y + c_3}, \quad c_2, c_3 \in \mathbb{R}, \quad c_2 \neq 0.$$

This implies the statement (i.3) of Theorem 1.1.

Case A.3.2. $c_1 \neq 0$. It follows from (3.3) that

$$(3.5) \quad c_1 \left((f'g)^2 + 1 \right) + f f'' - 2(f')^2 = 0.$$

Taking partial derivative of (3.5) with respect to z yields $g' = 0$ which is not possible because of the regularity.

Case B. $H_0 \neq 0$. We have cases:

Case B.1. $f = f_0 \neq 0 \in \mathbb{R}$. Then (3.1) follows

$$(3.6.) \quad 2H_0 f_0^2 = \frac{g''}{(g')^3},$$

and solving it gives $g(z) = \pm \frac{1}{2H_0 f_0^2} \sqrt{-4H_0 f_0^2 z + c_1} + c_2$, $c_1, c_2 \in \mathbb{R}$. This is the proof of the statement (ii) of Theorem 1.1.

Case B.2. $f = c_1 y + c_2$, $c_1, c_2 \in \mathbb{R}$, $c_1 \neq 0$. By considering this one in (3.1) we conclude

$$(3.7) \quad 2(c_1 y + c_2)^2 H_0 = (1 + c_1^2 g^2) \frac{g''}{(g')^3} - 2c_1^2 \frac{g}{g'}.$$

The left side in (3.7) is a function of y while other side is either a constant or a function z . This is not possible.

Case B.3. $f'' \neq 0$. By multiplying both side of (3.1) with $2f^2 \frac{g'}{g}$ one can be rearranged as

$$(3.8) \quad 2H_0 f^2 \frac{g'}{g} = \left((f'g)^2 + 1 \right) \frac{g''}{g(g')^2} + ff'' - 2(f')^2.$$

Taking partial derivative of (3.8) with respect to z and after dividing with $(f')^2$ yields

$$(3.9) \quad 2H_0 \left(\frac{f}{f'} \right)^2 \left(\frac{g'}{g} \right)' = 2 \frac{g''}{g'} + \left(g^2 + \frac{1}{(f')^2} \right) \left(\frac{g''}{g(g')^2} \right)'.$$

After again taking partial derivative of (3.8) with respect to y we have

$$(3.10) \quad 2H_0 \left(\left(\frac{f}{f'} \right)^2 \right)' \left(\frac{g'}{g} \right)' = \left(\frac{1}{(f')^2} \right)' \left(\frac{g''}{g(g')^2} \right)'.$$

In order to solve (3.10) we have to consider several cases:

Case B.3.1. $f' = c_1 f$, $c_1 \in \mathbb{R}$, $c_1 \neq 0$. (3.10) leads to the following:

Case B.3.1.1. $g'' = 0$, i.e, $g = c_2 z + c_3$, $c_2, c_3 \in \mathbb{R}$, $c_2 \neq 0$. Then (3.8) reduces to

$$(3.11) \quad 2H_0 \frac{c_2}{c_2 z + c_3} = -c_1^2,$$

which is a contradiction.

Case B.3.1.2. $g'' = c_2 g (g')^2$, $c_2 \in \mathbb{R}$, $c_2 \neq 0$. By dividing (3.8) with f^2 we get that

$$2H_0 \frac{g'}{g} = c_1^2 c_2 g^2 + \frac{c_2}{f^2} - c_1^2$$

and taking its partial derivative of y gives the contradiction $f' = 0$.

Case B.3.2. $f' \neq c_1 f$, $c_1 \in \mathbb{R}$. If $g' = c_2 g$, $c_2 \in \mathbb{R}$, $c_2 \neq 0$ in (3.10) then it follows

$$0 = \left(\frac{1}{(f')^2} \right)' \left(\frac{1}{g^2} \right)',$$

which is not possible since $f'' \neq 0$ and $g' \neq 0$. Hence (3.10) can be rewritten as

$$(3.12) \quad 2H_0 \frac{\left(\left(\frac{f}{f'} \right)^2 \right)'}{\left(\frac{1}{(f')^2} \right)'} = \frac{\left(\frac{g''}{g(g')^2} \right)'}{\left(\frac{g'}{g} \right)'}$$

Both sides in (3.12) have to be a nonzero constant c_3 . Thereby (3.12) yields that

$$(3.13) \quad \left(\frac{f}{f'} \right)^2 = \frac{c_3}{(f')^2} + c_4$$

and

$$(3.14) \quad \frac{g''}{g(g')^2} = 2H_0 c_3 \frac{g'}{g} + c_5,$$

where $c_4, c_5 \in \mathbb{R}$. The fact that f is a non-constant function leads to $c_4 \neq 0$. (3.13) implies

$$(3.15) \quad f'' = \frac{1}{c_4} f.$$

Considering (3.13) – (3.15) in (3.8) gives

$$(3.16) \quad 2H_0 f^2 \frac{g'}{g} = \left(\left(\frac{f^2 - c_3}{c_4} \right) g^2 + 1 \right) \left(2H_0 c_3 \frac{g'}{g} + c_5 \right) - \frac{f^2}{c_4} + \frac{2c_3}{c_4}.$$

By taking partial derivative of (3.16) with respect to y , we find

$$(3.17) \quad -2H_0 \frac{g'}{g} = \frac{c_5 g^2 - 1}{c_3 g^2 - c_4}.$$

Case B.3.2.1. $c_5 = 0$. Then (3.14) follows

$$(3.18) \quad g' = \frac{-1}{2H_0 c_3 g + c_6}, \quad c_6 \in \mathbb{R}.$$

Substituting (3.18) in (3.17) leads to

$$\frac{2H_0}{2H_0 c_3 g^2 + c_6 g} = \frac{-1}{c_3 g^2 - c_4}$$

or the following polynomial equation on g :

$$(3.19) \quad 4H_0 c_3 g^2 + c_6 g - 2H_0 = 0.$$

Since the coefficients H_0 and c_3 are nonzero, we obtain a contradiction.

Case B.3.2.2. $c_5 \neq 0$. Since $\frac{g'}{g}$ is not constant, (3.17) immediately implies $c_4 \neq \frac{c_3}{c_5}$. Substituting (3.17) into (3.14) gives

$$\frac{g''}{g'} = (c_3 - c_4 c_5) \frac{g g'}{c_3 g^2 - c_4}.$$

or

$$(3.20) \quad g' = c_7 (c_3 g^2 - c_4)^{\frac{c_3 - c_4 c_5}{2c_3}}, \quad c_7 \neq 0.$$

By considering (3.20) in (3.17) we deduce

$$(3.21) \quad -2H_0 c_6 (c_3 g^2 - c_4)^{\frac{3c_3 - c_4 c_5}{2c_3}} = c_5 g^3 - g.$$

This leads to a contradiction since the terms g of different degrees appears in (3.21).

4. PROOF OF THEOREM 1.2

By a calculation for a factorable graph of type 2 in \mathbb{T}^3 , the isotropic Gaussian curvature turns to

$$(4.1) \quad K = \frac{f g f'' g'' - (f' g')^2}{(f g')^4}.$$

Let us assume that $K = K_0 = \text{const.}$ We have cases:

Case A. $K_0 = 0$. (4.1) reduces to

$$(4.2) \quad f g f'' g'' - (f' g')^2 = 0.$$

f or g constants are solutions for (4.2) and by regularity we have the statement (i.1) of Theorem 1.2. Suppose that f, g are non-constants. Then (4.2) yields $f'' g'' \neq 0$. Thereby (4.2) can be arranged as

$$(4.3) \quad \frac{f f''}{(f')^2} = \frac{(g')^2}{g g''}.$$

Both sides of (4.3) are equal to same nonzero constant, namely

$$(4.4) \quad ff'' - c_1 (f')^2 = 0 \text{ and } gg'' - \frac{1}{c_1} (g')^2 = 0.$$

If $c_1 = 1$ in (4.4), then by solving it we obtain

$$f(y) = c_2 e^{c_3 y} \text{ and } g(z) = c_4 e^{c_5 z}, \quad c_2, \dots, c_5 \in \mathbb{R}.$$

This gives the statement (i.2) of Theorem 1.2. Otherwise, i.e. $c_1 \neq 1$ in (4.4), we derive

$$f(y) = ((1 - c_1)(c_6 y + c_7))^{\frac{1}{1-c_1}} \text{ and } g(z) = \left(\left(\frac{c_1 - 1}{c_1} \right) (c_8 z + c_9) \right)^{\frac{c_1}{c_1 - 1}},$$

where $c_6, \dots, c_9 \in \mathbb{R}$. This completes the proof of the statement (i) of Theorem 1.2.

Case B. $K_0 \neq 0$. (4.1) can be rewritten as

$$(4.5) \quad K_0 (g')^2 = \frac{f''}{f^3} \left(\frac{gg''}{(g')^2} \right) - \left(\frac{f'}{f^2} \right)^2.$$

Taking partial derivative of (4.5) with respect to z leads to

$$(4.6) \quad 2K_0 g' g'' = \frac{f''}{f^3} \left(\frac{gg''}{(g')^2} \right)'.$$

We have several cases for (4.6):

Case B.1. $g'' = 0$, $g(z) = c_1 z + c_2$, $c_1, c_2 \in \mathbb{R}$. Hence from (4.5) we deduce

$$K_0 (c_1)^2 = - \left(\frac{f'}{f^2} \right)^2,$$

which implies that K_0 is negative and

$$f(y) = \frac{1}{\pm c_1 \sqrt{-K_0} y + c_3}.$$

This proves the statement (ii.1) of Theorem 1.2.

Case B.2. $g'' \neq 0$. (4.6) immediately implies $f'' \neq 0$. Then taking partial derivative of (4.6) with respect to y gives

$$(4.7) \quad 0 = \left(\frac{f''}{f^3} \right)' \left(\frac{gg''}{(g')^2} \right)',$$

or

$$(4.8) \quad f'' = c_1 f^3, \quad c_1 \in \mathbb{R}.$$

By considering (4.8) in (4.5) we get

$$(4.9) \quad K_0 (g')^2 = c_1 \frac{gg''}{(g')^2} - \left(\frac{f'}{f^2} \right)^2.$$

Taking partial derivative of (4.9) with respect to y leads to

$$(4.10) \quad f' = c_2 f^2, \quad c_2 \in \mathbb{R}.$$

It follows from (4.8) and (4.10) that $c_1 = 2c_2^2$ and

$$f(y) = -\frac{1}{c_2 y + c_3}$$

for some constant c_3 . By substituting (4.8) and (4.10) into (4.5), we conclude

$$(4.11) \quad \frac{K_0}{c_2^2} r^3 + r = 2gr',$$

where $r = g'$ and $r' = \frac{dr}{dg} = \frac{g''}{g'}$. After solving (4.11), we obtain

$$r = \pm \left(c_4^2 g^{-1} - \frac{K_0}{c_2^2} \right)^{-1/2}, \quad c_4 \in \mathbb{R}, \quad c_4 \neq 0,$$

or

$$z = \pm \int \left(c_4^2 g^{-1} - \frac{K_0}{c_2^2} \right)^{1/2} dg,$$

which proves the statement (ii.2) of Theorem 1.2.

5. PROOF OF THEOREM 1.3

Assume that a factorable surface of type 2 in \mathbb{I}^3 fulfills the condition $H + \lambda K = 0$, $\lambda HK \neq 0$. Then (3.1) and (4.1) give rise to

$$(5.1) \quad f^2 \left((f'g)^2 + 1 \right) g'g'' + f^2 g (g')^3 \left(ff'' - 2(f')^2 \right) + 2\lambda \left(ff''gg'' - (f'g')^2 \right) = 0.$$

Since $K \neq 0$, (5.1) can be divided by $(ff')^2$ as follows:

$$(5.2) \quad \left(g^2 + \frac{1}{(f')^2} \right) g'g'' + g(g')^3 \left(\frac{ff''}{(f')^2} - 2 \right) + 2\lambda \frac{ff''}{f(f')^2} gg'' - \frac{2\lambda}{f^2} (g')^2 = 0.$$

In order to solve (5.2) we have to distinguish several cases:

Case A. $g = c_1 z + c_2$, $c_1, c_2 \in \mathbb{R}$, $c_1 \neq 0$. (5.2) reduces to

$$(5.3) \quad c_1 (c_1 z + c_2) \left(\frac{ff''}{(f')^2} - 2 \right) - \frac{2\lambda}{f^2} = 0.$$

Taking partial derivative of (5.3) with respect to z gives $ff'' = 2(f')^2$. Considering it into (5.3) yields the contradiction $\lambda = 0$.

Case B. $g'' \neq 0$. By dividing (5.2) with the product $g'g''$, we get

$$(5.4) \quad g^2 + \frac{1}{(f')^2} + \frac{g(g')^2}{g''} \left(\frac{ff''}{(f')^2} - 2 \right) + 2\lambda \frac{ff''g}{f(f')^2 g'} - \frac{2\lambda g'}{f^2 g''} = 0.$$

Put $p = f'$, $p' = \frac{dp}{df} = \frac{f''}{f'}$, $r = g'$ and $r' = \frac{dr}{dg} = \frac{g''}{g'}$ in (5.4). Thus taking partial derivatives of (5.4) with respect to f and g implies

$$(5.5) \quad \left(\frac{gr}{r'} \right)' \left(\frac{fp'}{p} \right)' + 2\lambda \left(\frac{p'}{fp} \right)' \left(\frac{g}{r} \right)' - 2\lambda \left(\frac{1}{f^2} \right)' \left(\frac{1}{r'} \right)' = 0.$$

Since $(1/f^2)' \neq 0$, we can rewrite (5.5) as

$$(5.6) \quad \left(\frac{gr}{r'} \right)' \frac{(fp'/p)'}{(1/f^2)'} + 2\lambda \left(\frac{g}{r} \right)' \frac{(p'/fp)'}{(1/f^2)'} - 2\lambda \left(\frac{1}{r'} \right)' = 0.$$

Taking derivative of (5.6) with respect to f leads to

$$(5.7) \quad \left(\frac{gr}{r'} \right)' \left(\frac{(fp'/p)'}{(1/f^2)'} \right)' + 2\lambda \left(\frac{g}{r} \right)' \left(\frac{(p'/fp)'}{(1/f^2)'} \right)' = 0.$$

We have some cases to solve (5.7):

Case B.1. $p' = 0$, i.e. $f(y) = c_1 y + c_2$, $c_1, c_2 \in \mathbb{R}$, $c_1 \neq 0$. Considering it into (5.4) gives

$$(5.8) \quad g^2 + \frac{1}{c_1^2} - 2 \frac{g(g')^2}{g''} - \frac{2\lambda g'}{f^2 g''} = 0$$

and taking partial derivative of (5.8) with respect to y implies that f' or g' vanish however both situations are not possible.

Case B.2 In (5.7) assume that $p' \neq 0$ and

$$(5.9) \quad \left(\frac{(fp'/p)'}{(1/f^2)'} \right)' = \left(\frac{(p'/fp)'}{(1/f^2)'} \right)' = 0.$$

This one follows

$$(5.10) \quad \frac{fp'}{p} = \frac{c_1}{f^2} + c_2 \text{ and } \frac{p'}{fp} = \frac{c_3}{f^2} + c_4, \quad c_1, \dots, c_4 \in \mathbb{R}.$$

Both equalities in (5.10) imply

$$(5.11) \quad \frac{p'}{p} = \frac{c_2}{f^2}, \quad c_2 \neq 0, \quad c_2 = c_3.$$

Considering (5.11) in the first or second equality of (5.9) leads to a contradiction.

Case B.3 $(g/r)' = 0$. This implies $g' = c_1 g$, namely $g = c_2 e^{c_1 z}$, $c_1, c_2 \in \mathbb{R}$, $c_1 c_2 \neq 0$. Substituting it into (5.4) yields

$$f(y) = c_3 e^{c_4 y},$$

which is a contradiction since $K \neq 0$.

Case B.4. $(gr/r')' = 0$ in (5.7). Then $r' = c_1 gr$, $c_1 \in \mathbb{R}$, $c_1 \neq 0$, and (5.6) reduces to

$$(5.12) \quad \left(\frac{g}{r} \right)' \frac{(p'/fp)'}{(1/f^2)'} - \left(\frac{1}{r'} \right)' = 0,$$

and taking partial derivative of (5.12) with respect to g yields

$$(5.13) \quad \frac{(p'/fp)'}{(1/f^2)'} = c_2 \neq 0.$$

Substituting (5.13) into (5.12) gives

$$(5.14) \quad r = c_3 g - \frac{1}{c_4 g}, \quad c_3, c_4 \in \mathbb{R}, \quad c_3 c_4 \neq 0.$$

By taking derivative of (5.14) with respect to g and comparing with $r' = c_1 gr$, we deduce the following polynomial equation on g :

$$c_1 c_3 g^4 - \left(\frac{c_1}{c_4} + c_3 \right) g^2 - \frac{1}{c_2} = 0.$$

This gives a contradiction.

Case B.5. $(gr/r')' \neq 0$ in (5.8). Then (5.8) can be rewritten as

$$\underbrace{\left(\left(\frac{g}{r} \right)' \right)^{-1} \left(\frac{gr}{r'} \right)'}_{G(g)} + 2\lambda \underbrace{\left(\frac{(p'/fp)'}{(1/f^2)'} \right)' \left(\left(\frac{(fp'/p)'}{(1/f^2)'} \right)' \right)^{-1}}_{F(f)} = 0,$$

which implies that $G(g) = c_1$, $F(f) = -c_1/2\lambda$, $c_1 \in \mathbb{R}$, $c_1 \neq 0$. Thus we have

$$(5.15) \quad \frac{gr}{r'} = c_1 \frac{g}{r} + c_2, \quad c_2 \in \mathbb{R}.$$

Substituting (5.15) in (5.5) follows

$$(5.16) \quad c_1 \left(\frac{fp'}{p} \right)' + 2\lambda \left(\frac{p'}{fp} \right)' - 2\lambda \left(\frac{1}{f^2} \right) \frac{(1/r')'}{(g/r)'} = 0.$$

By taking partial derivative of (5.16) with respect to g , we find

$$(5.17) \quad \frac{g}{r} = \frac{c_3}{r'} + c_4, \quad c_3, c_4 \in \mathbb{R}, \quad c_4 \neq 0.$$

Substituting (5.15) and (5.17) into (5.4) gives

$$(5.18) \quad \frac{g}{r} = c_5 g^2 + c_6, \quad c_5, c_6 \in \mathbb{R}, \quad c_5 \neq 0.$$

(5.15), (5.17) and (5.18) imply the following polynomial equation:

$$(c_1 c_5 - c_3 c_5^2) g^4 + (c_1 c_6 - 2c_3 c_5 c_6 - (c_2 - c_4) c_5) g^2 + (c_2 - c_4) c_6 - c_3 c_6^2 = 0,$$

which yields $c_1 = c_3 c_5$, $c_2 = c_4$, $c_6 = 0$. Hence we get from (5.18)

$$(5.19) \quad gg' = c_7, \quad c_7 \in \mathbb{R}, \quad c_7 \neq 0.$$

Substituting (5.19) into (5.4) yields

$$(5.20) \quad g^2 + \frac{1}{(f')^2} + g^2 \left(\frac{ff''}{(f')^2} - 2 \right) + \frac{2\lambda g^2}{c_7} \left(\frac{f''}{f(f')^2} \right) + \frac{2\lambda g^2}{c_7 f^2} = 0.$$

By dividing (5.20) with g^2 and after taking derivative with respect to z , we obtain the contradiction $g' = 0$.

6. SOME EXAMPLES

We illustrate some examples related with constant curvature factorable surfaces of type 2 in \mathbb{I}^3 .

Example 6.1. Consider the factorable surfaces of type 2 in \mathbb{I}^3 given by

- (1) $\Phi_3 : x = y \tan z$, $(y, z) \in [0, \frac{\pi}{3}]$, (isotropic minimal),
- (2) $\Phi_3 : x = -\sqrt{z}$, $(y, z) \in [0, 2\pi]$, ($H = -1$),
- (3) $\Phi_3 : x = -\frac{y^2}{4z}$, $(y, z) \in [1, 1.4] \times [1, 2\pi]$, (isotropic flat),
- (4) $\Phi_3 : x = \frac{z}{y}$, $(y, z) \in [1, \pi] \times [1, 2\pi]$, ($K = -1$).

The surfaces can be respectively plotted by Wolfram Mathematica 7.0 as in Fig.1, ..., Fig.4.

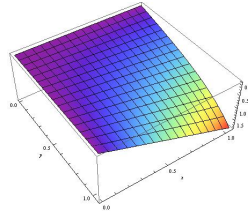


FIGURE 1. An isotropic minimal factorable surface of type 2.

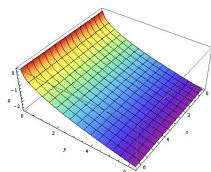


FIGURE 2. A factorable surface of type 2 with $H = -1$.

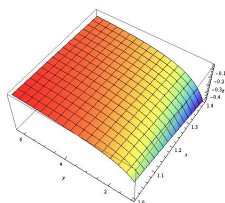


FIGURE 3. An isotropic flat factorable surface of type 2.

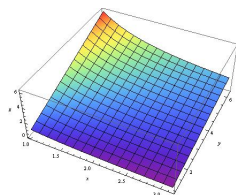


FIGURE 4. A factorable surface of type 2 with $K = -1$.

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